

Flow Phase Diagram for the Helium Superfluids

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The flow phase diagram for He II and $^3\text{He-B}$ is established and discussed based on available experimental data and the theory of Volovik [JETP Letters **78** (2003) 553]. The effective temperature - dependent but scale - independent Reynolds number $Re_{eff} = 1/q = (1 + \alpha')/\alpha$, where α and α' are the mutual friction parameters and the superfluid Reynolds number characterizing the circulation of the superfluid component in units of the circulation quantum are used as the dynamic parameters. In particular, the flow diagram allows identification of experimentally observed turbulent states I and II in counterflowing He II with the turbulent regimes suggested by Volovik.

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We consider the flow of quantum liquids such as He II or $^3\text{He-B}$ that can be described in the framework of the two fluid model (see, e.g., [1]). The normal fluid and superfluid velocity fields are coupled by two terms - by the Gorter- Mellink term that describes the mutual friction between these two liquids when vortices are present in the superfluid, and by the temperature gradient term, responsible, e.g., for the fountain effect. Circulation in the superfluid component is quantized in units of κ ($0.997 \times 10^{-3} \text{ cm}^2/\text{s}$ for He II and $0.662 \times 10^{-3} \text{ cm}^2/\text{s}$ for $^3\text{He-B}$); we assume singly quantized vortices.

Let us consider a flow that can be approximated as isothermal [2]. Then the generally coupled complex flow of both components described by the two two-fluid equations can be simplified and becomes easier to understand, especially for two extreme cases:

i) *There are no quantized vortices in the flow.* This represents a situation when the normal and superfluid velocity fields are fully decoupled. The normal fluid thus obeys the usual Navier-Stokes equation (and standard fluid dynamics) while the superfluid flow stays potential. Thus formally the normal fluid could become turbulent without a single vortex being present in the superfluid - in the absence of mutual friction the superfluid simply does not know what is happening in the normal fluid. In practice, however, remnant vortices are almost always present, at least in He II [3], pinned to walls which are always rough on the atomic scale. In $^3\text{He-B}$ a vortex free sample is more likely, but the highly viscous normal fluid can hardly become turbulent in the laboratory sized container.

ii) *The normal fluid is at rest in some frame of reference.* Such a possibility arises, for example, for $^3\text{He-B}$, whose highly viscous normal component is effectively clamped by the walls in a laboratory size container. Moreover, we assume that the quantized vortices in the flow are arranged in such a way that the coarse-grained hydrodynamic equation

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \mu = (1 - \alpha') \mathbf{v}_s \times \boldsymbol{\omega} + \alpha \hat{\boldsymbol{\omega}} \times (\boldsymbol{\omega} \times \mathbf{v}_s) \quad (1)$$

obtained from the Euler equation after averaging over vortex lines [4], written in the frame of reference of the normal fluid, provides a sufficiently accurate description of the superflow. We shall return to the applicability of this equation later. The normal fluid thus provides a unique frame of reference and we have to deal only with the superfluid velocity \mathbf{v} . By re-scaling the time such that $\tilde{t} = (1 - \alpha')t$, when dropping the tilde sign, one gets

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \mu = \mathbf{v} \times \boldsymbol{\omega} + q \hat{\boldsymbol{\omega}} \times (\boldsymbol{\omega} \times \mathbf{v}) \quad (2)$$

The theoretical analysis of the fluid dynamical problem based on this equation has been performed by Volovik [5]. As was first emphasized by Finne *et al* [6], it has a very remarkable property which makes it distinct from the ordinary Navier-Stokes equation where the relative importance of the inertial and dissipative terms is given by the Reynolds number, which in turn depends on the geometry of the particular flow under study. Here the role of the effective Reynolds number is played by the parameter $Re_{eff} = q^{-1} = (1 + \alpha')/\alpha$ that depends on temperature but not on geometry. We stress that the superfluid Reynolds number is not relevant to consideration of the problem of flow obeying eq.(2), the beauty of which consists in the fact that one is able to derive more general conclusions about turbulent flow generated from suitable initial conditions depending only on a single temperature dependent parameter $1/q$, regardless of the actual geometry of the flow. A wide range of q values is easily experimentally achievable; with q increasing with temperature in both He II [7] and $^3\text{He-B}$ [8].

Like the usual Navier-Stokes equation, eq.(2) is a non-linear differential equation allowing for both stable and unstable solutions including fully developed turbulence. For $q \gg 1$, similar to the low Re classical fluid dynamics, the solutions are stable. As q approaches unity, solutions become unstable and for $q \ll 1$ it describes fully developed turbulence. The latter is discussed in detail in [5], showing that a turbulent cascade will develop, covering scales from the outer scale of the order of the container

size, R , down to a minimum scale $r_o \sim q^{3/2}R$ with the velocity scale $v_{r0} \sim q^{1/2}U$, U being the velocity at the outer scale R . The 3D energy spectrum remains in its usual Kolmogorov form $E(k) \cong \varepsilon^{2/3}k^{-5/3}$ and the usual relation for the energy decay rate $\varepsilon = -dE/dt = v_r^3/r = U^3/R$ also holds in the $q \ll 1$ limit. The dissipation mechanism is, however, different in that instead $\varepsilon = \nu\omega^2$ for classical turbulence it now depends explicitly on the large scale velocity: $\varepsilon = -dE/dt \sim q\omega U^2$.

It is even possible [9] to work out logarithmic corrections to this approach, assuming that at each scale there is a direct transfer of kinetic energy to the normal component. In this refined approximation, the functional form of $E(k)$ becomes slightly modified by the logarithmic correction, but the roll-off exponent $-5/3$ remains unchanged. This analysis is closely connected with problems of fully developed turbulence in classical liquids, where similar logarithmic corrections have been proposed by various authors. It should therefore be of interest to the classical turbulence community, but details are beyond the scope of this Letter.

The continuous approach for considering superfluid turbulence based on eq.(2) is fully applicable in the limit $\kappa \rightarrow 0$. As pointed out by Volovik [5], at finite κ one has to ensure that, at the smallest scale r_0 , the "granularity" due to individual vortices does not become important, so that the circulation $v_{r0}r_0 = q^2UR = q^2\kappa Re_s > \kappa$. This leads to an important criterion $Re_s > 1/q^2 \gg 1$. For small enough q , when the turbulent cascade reaches small scales that would contain only a few quantized vortices, the turbulent cascade will most likely continue, but the form of the energy spectrum around and beyond the quantum scale [10], $\ell_q \approx (\varepsilon/\kappa^3)^{-1/4}$ must depend explicitly on κ [11].

In order to apply an analysis based on eq.(2), we must bear in mind that this coarse-grained equation sufficiently accurately describes the superfluid velocity field on the scale over which the averaging is done. This approach cannot therefore include initial conditions similar to those commonly believed to apply in counterflow turbulence in He II if only a single scale is assumed. Such a distribution of vortices will most likely decay according to the Vinen [12] equation. It is well known and agrees with simulations by Schwarz [13] that there is a critical self-sustaining counterflow velocity, above which the turbulence is in dynamical equilibrium. Let us call it turbulent state I, in accord with Tough's classification scheme [1]. According to computer simulations and a common belief based on the experiments of Awschalom *et al* [14] this state is, at least approximately, homogeneous. If it contains just one scale, the vortex line density, L , ought to decay as $1/t$ as follows from the Vinen equation and, according to some experiments [12, 15], it does. Now let increase the mean flow (i.e., the counterflow velocity U_{cf}), assuming the normal fluid velocity profile remains flat, and continue discussion in the reference frame where

the normal fluid is at rest. It is an established experimental fact that the transition to state II occurs [1, 16, 17], with distinctly different features. It has been a long lasting challenge to explain the nature of this transition. We believe that the answer is hidden in Volovik's analysis [5]. As he shows, there is a crossover between what he calls the Kolmogorov and Vinen states of superfluid turbulence when

$$Re_sq^2 = U_{cf}Rq^2/\kappa \simeq 1 \quad (3)$$

For higher counterflow velocities an analysis based on equation (2) is valid and therefore a range of scales between the outer scale, R and a scale on which the circulation is of order κ (the quantum scale) occurs. Well above this transition, (i.e., for high enough Re_s) there will be eddies spanning a wide range of scales and within all these scales the counterflow turbulence in the superfluid is of the Kolmogorov type.

But now that we have large superfluid eddies up to the size of the channel, these will interact via mutual friction with the normal fluid. Therefore the normal fluid is most likely driven into a turbulent state, too, so that the high Re_s counterflow turbulence should be similar in character to the grid generated turbulence. It is especially true for the case when the heater is turned off and the turbulence decays. We believe that this is the reason why an anomalous decay of counterflow turbulence in He II was observed in the pioneering work of Vinen [12] and later by Schwarz and Rosen [19]. In our own decay experiments [20], we have observed that the temperature gradient along the counterflow channel decays very fast when the heater is switched off, so the flow can be considered as isothermal when the second sound decay measurement is being performed. We therefore expect that the decays of high Re_s counterflow turbulence and grid generated He II turbulence ought to display the same character. And, indeed, it was clearly shown in experiments, that for both towed grid generated He II turbulence [21] and high Re_s counterflow turbulence [20], most of the decay of the vortex line density displays the same $t^{-3/2}$ power law. This decay law follows from the spectral decay model [21] and is based on the existence of the Kolmogorov form of the 3d energy spectrum, directly shown to be present in classically generated turbulence both above and below the lambda point by Maurer and Tabeling [22].

That the normal fluid is turbulent in state II of the counterflow He II turbulence is independently supported by the stability analysis of Melotte and Barenghi [23].

The crossover to superfluid turbulence state II has been observed in channels of circular and square cross-section, but not in channels of high aspect ratio rectangular cross-section [24]. Naturally, the transition cannot take place if the size of the sample intervenes. If some dimension of the channel is too small, its physical size R_0 limits the size of possible eddies.

It has been stated many times (see, e.g., [25]) that counterflow turbulence does not have any classical analog. On the other hand, a close similarity between counterflow turbulence and turbulent thermal convection in the heat transport efficiency has been pointed out [20]. It seems now that both views are justified. Counterflow turbulence in state I exists owing to the finite value of the circulation quantum and therefore cannot have any classical analog. However, the counterflow turbulence in state II is closely similar to turbulent thermal convection in the same way as the grid generated classical and He II turbulence closely resemble each other [18, 21].

There are many experimental data that can be used in order to verify, at least qualitatively, the phase diagram (see Fig.1) suggested by Volovik [5]. The recent experiment of Finne *et al* [6] provides evidence for a velocity-independent transition from a laminar to a turbulent flow regime in rotating $^3\text{He-B}$, where values of q of order unity are experimentally easily accessible. In He II these large values of q occur very close to the lambda point, where, to our knowledge, no reliable measurements exist that can be considered in the frame of reference of the normal fluid. On the other hand, there is ample experimental data on counterflow He II turbulence at lower temperatures. However, the data on the transition into superfluid state I (Vinen state) in tubes and capillaries of various sizes cannot be reliably used here, as it is believed that below this threshold the viscous normal fluid possesses a velocity profile similar to a flow of ordinary viscous flow in a pipe. A unique frame of reference in not, therefore, provided by the normal fluid. However, Baehr *et al* [26] studied the transition from dissipationless superflow to homogeneous superfluid turbulence, when both ends of the pipe were blocked by superleaks and the normal fluid inside the pipe thus remains stationary, thereby providing this unique frame of reference. These data, spanning the temperature range $1.3\text{ K} < T < 1.9\text{ K}$, mark the transition from laminar flow into the Vinen state (state I) and are shown in Fig. 1.

Various counterflow experiments clearly display the transition from state I (Vinen) into state II (Kolmogorov) - the signature is pronounced on temperature and pressure difference versus heat input dependencies. We use here the data of Ladner, Childers and Tough [16] (their Table I), assuming that in state I the normal fluid profile is flat, again providing the unique frame of reference with the normal fluid at rest. Let us point out that this transition into a different flow regime is accompanied with a pronounced increase of fluctuations [17], characteristic of phase transitions. The data [16] also clearly show that on increasing the temperature the difference in counterflow velocity between state I and II transitions decreases and around 2 K they become indistinguishable.

As another set of experimental data marking the state I-state II transition we have used the thermal conduction measurements of He II in tubes of various diameters of

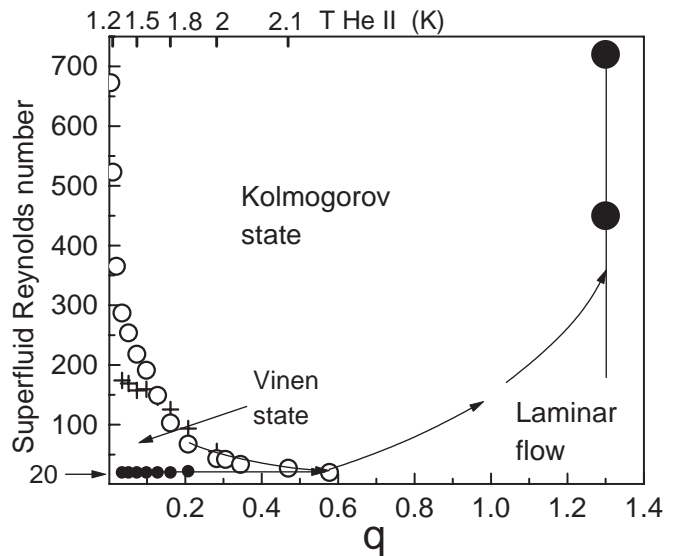


FIG. 1: The observed flow diagram of He II and $^3\text{He-B}$ superfluids in the unique frame of reference where the normal fluid is at rest. The abscissa, q , represents the inverse of the Reynolds number for superflow (for convenience the corresponding temperature in He II is indicated on the upper axis), while the ordinate, Re_s , represents the strength of circulation at the outer scale of the flow in units of κ . The small filled circles represent the onset of turbulent state I in pure superflow of He II when the motion of the normal fluid was inhibited by superleaks [26]; the crosses [16] and open circles [27] mark the transition from state I into state II for counterflowing He II. The two big filled points mark approximately the region where the onset of superfluid turbulence has been observed by various methods of vortex loop injection into rotating $^3\text{He-B}$ in the vortex-free Landau state (for $\Delta T \approx 0.05 T_c$ around $0.6 T/T_c$ at 29 bar, see Fig.3 in [6])

Chase [27]. We have scanned the available experimental data and show in Fig. 2 that they collapse onto a single curve if Reynolds number scaling is applied. The open circles in Fig. 1 correspond to the onset of state II.

We emphasize that the procedure used to acquire the data points shown in the flow diagram is probably not very accurate for several reasons, such as different temperature scales or uncertainty in values of q , and more work is needed to map it out accurately. We believe, however, that the essential physics is displayed clearly and that Fig.1 strongly supports the ideas underlying the physical problem of superfluid turbulence.

Let us comment here on the apparent disagreement between this phase diagram and the computer simulations of Araki *et al* [28] done in the zero temperature limit in the sense that there is no normal fluid, strongly indicating the presence of Kolmogorov scaling. This computer simulation introduces a cutoff at the scale of the grid used for simulations, in that all vortex rings or loops smaller than this size are removed from the flow. This effectively introduces an artificial dissipative mechanism at a

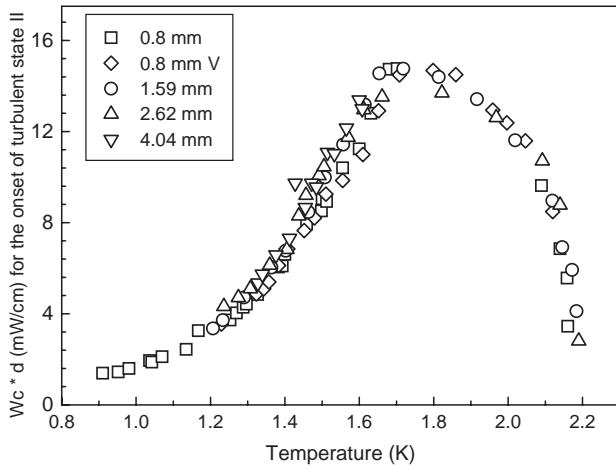


FIG. 2: Product of the critical heat input per unit area, Wc , times the inner diameter of the used tube, d , marking the onset of the turbulent state II versus temperature [27] (two different experimental methods have been used for a tube with $d = 0.8$ mm). The data obtained with tubes of various d as indicated collapse onto a single curve. Assuming that $U_{cf} \propto Wc$, the onset of turbulent state II occurs at any temperature at a critical value of Re_s , shown in Fig.1.

prescribed length scale. We believe that any dissipative mechanism acting at some small length scale (such as a Kelvin wave cascade with subsequent phonon emission [18]) leads to a Kolmogorov cascade in the continuum approximation, as the assumptions for it are only that there is a range of scales where dissipation is unimportant, that the form of the energy spectrum only depends on k , and that the total energy decay rate $\varepsilon = -dE/dt$ is independent of k . Dimensional analysis then leads to the energy spectrum of the Kolmogorov form. The physical mechanism of the dissipation is unimportant, so long as it acts only on small scales.

In practice, there should be a crossover from the mutual friction dissipation regime correctly described by Volovik's analysis into a different one, probably based on vortex wave irradiation. The governing equation (2) will have to be altered accordingly, and similar analysis as in [5] ought to be repeated. A Kolmogorov cascade will most likely emerge again, as soon as the smallest scale obtained with this new dissipation mechanism is larger than the quantum scale.

To conclude, we show that the extraordinary fluid properties of quantum fluids give rise to the flow diagram suggested by Volovik [5], containing two distinctly different turbulent flow regimes called Vinen and Kolmogorov regimes. These can most likely be identified with the puzzling turbulent states I and II according to the classification scheme of Tough [1].

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